

One-way Quantum Deficit and Decoherence for Two-qubit X States

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We study one-way quantum deficit of two-qubit X states systematically from analytical derivations. An effective approach to compute one-way quantum deficit of two-qubit X states has been provided. Analytical results are presented as for detailed examples. Moreover, we demonstrate the decoherence of one-way quantum deficit under phase damping channel.

Keywords One-way quantum deficit; two-qubit X states; decoherence

I. INTRODUCTION

Quantum entanglement is one of the most distinguishing properties in quantum mechanics, which gives quantum information processing novel advantages over classical information processing [1, 2]. Recently, quantum correlations [3] receive much attention because they may play vital roles in quantum information processing and quantum simulation even without quantum entanglement [4, 5]. The characterization and quantification of quantum correlations have become one of the significant topics in the past decade [3]. One of the quantum correlations is characterized by quantum discord [6, 7] which is shown to play important roles in quantum information tasks such as quantum state discrimination [8, 9], remote state preparation [10] and quantum state merging [11, 12]. There have been many kinds of quantum correlations like measurement-induced disturbance [13], geometric quantum discord [14, 15], relative entropy of discord [16], continuous-variable discord [17, 18] etc.. However, analytical computation of these quantum correlations seem extremely difficult as optimization involved. Few analytical results have been obtained even for general two-qubit states. An analytical formula of quantum discord for Bell-diagonal states is provided in [19]. For general two-qubit X states, the analytical formula is still missed [20–24]. Recently, the authors in [25–27] presented a better classification in deriving analytical quantum discord for five parameters X states.

Among other quantum correlations, the quantum deficit is related to extract work from a correlated system coupled to a heat bath under nonlocal operations [28], with the work deficit defined by $W_t - W_l$, where W_t is the information of the whole system and W_l is the localizable

information [29]. It is also equal to the difference of the mutual information and the classical deficit [30]. The analytical formula of one-way quantum deficit [28, 29], like quantum discord, remains unknown. In Ref.[31] the quantum deficit of four-parameter two-qubit X states has been calculated. Numerical results for five-parameter X states are presented in Ref.[32]. In this paper, by using different approaches, we systematically compute the one-way quantum deficit for general two-qubit X states in terms of analytical derivations. Analytical results are presented for classes of detailed quantum states. Decoherence of one-way quantum deficit under phase damping channels is calculated too.

II. ONE-WAY QUANTUM DEFICIT OF TWO-QUBIT X STATES

We consider two-qubit states ρ_{AB} in Hilbert space $\mathcal{H}_A^2 \otimes \mathcal{H}_B^2$. The one-way quantum deficit [33] is defined according to the minimal increase of entropy after measurement on B ,

$$\tilde{\Delta}(\rho_{AB}) = \min_{M_k} S(\sum_k M_k \rho_{AB} M_k) - S(\rho_{AB}), \quad (1)$$

where the minimum is taken over all measurement operators $\{M_k\}$ satisfying $\sum_k M_k = \mathbb{1}$, $S(\rho) = -\text{Tr} \rho \log_2 \rho$ is von Neumann entropy. It is equal to the thermal discord [34]. It is also denoted by the relative entropy to the set of classical-quantum states [29].

Since the quantum correlations between A and B do not change under the local unitary operations, we consider ρ_{AB} in the Bloch representation as

$$\rho_{AB} = \frac{1}{4}[I \otimes I + x\sigma_3 \otimes I + yI \otimes \sigma_3 + \sum_{i=1}^3 t_i \sigma_i \otimes \sigma_i], \quad (2)$$

where σ_i ($i = 1, 2, 3$) are the Pauli matrices, x, y, t_1, t_2 and t_3 are real number. Equivalently under the computational bases $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$,

$$\rho_{AB} = \begin{pmatrix} a & 0 & 0 & f \\ 0 & b & e & 0 \\ 0 & e & c & 0 \\ f & 0 & 0 & d \end{pmatrix}, \quad (3)$$

where

$$a = \frac{1}{4}(1 + t_3 + x + y); \quad b = \frac{1}{4}(1 - t_3 + x - y); \quad (4)$$

$$c = \frac{1}{4}(1 - t_3 - x + y); \quad d = \frac{1}{4}(1 + t_3 - x - y); \quad (5)$$

$$e = \frac{1}{4}(t_1 + t_2); \quad f = \frac{1}{4}(t_1 - t_2). \quad (6)$$

ρ_{AB} is called two-qubit X state, in which the parameters satisfy the relations $a, b, c, d, e, f \geq 0$, $a + b + c + d = 1$, $|e|^2 \leq bc$ and $|f|^2 \leq ad$.

It is easily verified that $S(\rho_{AB})$ in (1) is given by $S(\rho_{AB}) = -\sum_{\pm}(u_{\pm} \log_2 u_{\pm} + v_{\pm} \log_2 v_{\pm})$, with

$$\begin{aligned} u_{\pm} &= \frac{1}{4}(1 + t_3 \pm \sqrt{(x + y)^2 + (t_1 - t_2)^2}), \\ v_{\pm} &= \frac{1}{4}(1 - t_3 \pm \sqrt{(x - y)^2 + (t_1 + t_2)^2}). \end{aligned} \quad (7)$$

For arbitrary rank-two two-qubit states, the projective measurements are optimal to minimize the von Neumann entropy [35]. They are also almost sufficient rank-three and four two-qubit states [36]. In the following, we focus on projective measurements. Let M_k be the local measurement operators on subsystem B , $M_k = V\Pi_k V^\dagger$, with $\Pi_k = |k\rangle\langle k|$, $k = 0, 1$, and $V \in U(2)$ unitary matrices. Generally V can be expressed as $V = tI + i\vec{y} \cdot \vec{\sigma}$, where $t \in R$ and $\vec{y} = (y_1, y_2, y_3) \in R^3$ satisfy $y_1^2 + y_2^2 + y_3^2 + t^2 = 1$. After measurement the state ρ_{AB} is transformed to the ensemble $\{\rho_k, p_k\}$ with

$$\rho_k = \frac{1}{p_k}(I \otimes M_k) \rho_{AB} (I \otimes M_k), \quad (8)$$

and $p_k = \text{tr}(I \otimes M_k) \rho_{AB} (I \otimes M_k)$. By tedious calculation [32], we have $\sum_k (I \otimes M_k) \rho_{AB} (I \otimes M_k) = p_0 \rho_0 + p_1 \rho_1$, where

$$p_0 \rho_0 = \frac{1}{4}[I + yz_3 I + t_1 z_1 \sigma_1 + t_2 z_2 \sigma_2 + (x + t_3 z_3) \sigma_3] \otimes V \Pi_0 V^\dagger, \quad (9)$$

$$p_1 \rho_1 = \frac{1}{4}[I - yz_3 I - t_1 z_1 \sigma_1 - t_2 z_2 \sigma_2 - (x - t_3 z_3) \sigma_3] \otimes V \Pi_1 V^\dagger, \quad (10)$$

$$z_1 = 2(-ty_2 + y_1 y_3), \quad z_2 = 2(ty_1 + y_2 y_3), \quad z_3 = t^2 + y_3^2 - y_1^2 - y_2^2.$$

From the matrix diagonalization techniques in Ref. [31], we have the eigenvalues of $\sum_k (I \otimes M_k) \rho_{AB} (I \otimes M_k)$,

$$\begin{aligned} \lambda_{1,2} &= \frac{1}{4} \left(1 + yz_3 \pm \sqrt{(x + t_3 z_3)^2 + t_1^2 z_1^2 + t_2^2 z_2^2} \right), \\ \lambda_{3,4} &= \frac{1}{4} \left(1 - yz_3 \pm \sqrt{(x - t_3 z_3)^2 + t_1^2 z_1^2 + t_2^2 z_2^2} \right), \end{aligned} \quad (11)$$

with $z_1^2 + z_2^2 + z_3^2 = 1$. Therefore the one-way quantum deficit (1) is given by

$$\tilde{\Delta} = \sum_{\pm} (u_{\pm} \log_2 u_{\pm} + v_{\pm} \log_2 v_{\pm}) + \min \left[-\sum_{i=1}^4 \lambda_i \log_2 \lambda_i \right]. \quad (12)$$

To find the analytical solutions of (12), let us set $z_1 = \sin \theta \cos \varphi$, $z_2 = \sin \theta \sin \varphi$ and $z_3 = \cos \theta$. Then

$$\lambda_{1,2} = \frac{1}{4} \left(p_+ \pm \sqrt{R + S_+} \right), \quad \lambda_{3,4} = \frac{1}{4} \left(p_- \pm \sqrt{R + S_-} \right), \quad (13)$$

where

$$p_{\pm} = 1 \pm y \cos \theta, \quad R = [t_1^2 \cos^2 \varphi + t_2^2 \sin^2 \varphi] \sin^2 \theta, \quad S_{\pm} = (x \pm t_3 \cos \theta)^2.$$

Because λ_i is the eigenvalue of the density matrix, $\lambda_i \geq 0$. Hence $p_{\pm} \geq \sqrt{R + S_{\pm}} \geq 0$.

Denote $G(\theta, \varphi) = -\sum_{i=1}^4 \lambda_i \log_2 \lambda_i$. The one-way quantum deficit is given by the minimal value of $G(\theta, \varphi)$. We observe that $G(\theta, \varphi) = G(\pi - \theta, \varphi)$ and $G(\theta, \varphi) = G(\theta, 2\pi - \varphi)$. Moreover, $G(\theta, \varphi)$ is symmetric with respect to $\theta = \pi/2$ and $\varphi = \pi$. Therefore, we only need to consider the case of $\theta \in [0, \pi/2]$ and $\varphi \in [0, \pi)$.

The extreme points of $G(\theta, \varphi)$ are determined by the first partial derivatives of G with respect to θ and φ ,

$$\frac{\partial G}{\partial \theta} = -\frac{\sin \theta}{4} H_{\theta}, \quad (14)$$

with

$$\begin{aligned} H_{\theta} = & \frac{R \csc \theta \cot \theta - t_3 \sqrt{S_+}}{\sqrt{R + S_+}} \log_2 \frac{p_+ + \sqrt{R + S_+}}{p_+ - \sqrt{R + S_+}} + y \log_2 \frac{p_-^2 - (R + S_-)}{p_+^2 - (R + S_+)} \\ & + \frac{R \csc \theta \cot \theta + t_3 \sqrt{S_-}}{\sqrt{R + S_-}} \log_2 \frac{p_- + \sqrt{R + S_-}}{p_- - \sqrt{R + S_-}}, \end{aligned} \quad (15)$$

and

$$\frac{\partial G}{\partial \varphi} = 2ef \sin^2 \theta \sin 2\varphi H_{\varphi}, \quad (16)$$

with

$$H_{\varphi} = \frac{1}{\sqrt{R + S_+}} \log_2 \frac{p_+ + \sqrt{R + S_+}}{p_+ - \sqrt{R + S_+}} + \frac{1}{\sqrt{R + S_-}} \log_2 \frac{p_- + \sqrt{R + S_-}}{p_- - \sqrt{R + S_-}}. \quad (17)$$

Since H_{φ} is always positive, $\frac{\partial G}{\partial \varphi} = 0$ implies that either $\sin 2\varphi = 0$, namely, $\varphi = 0, \pi/2$, for any θ , or $\theta = 0$ for any φ which implies that (14) is zero at the same time and the minimization is independent on φ . If $\theta \neq 0$, one gets the second derivative $\partial^2 G / \partial \varphi^2$

$$\frac{\partial^2 G}{\partial \varphi^2} \Big|_{(\theta, 0)} = 4ef \sin^2(\theta) H_{\varphi=0} > 0, \quad (18)$$

and

$$\frac{\partial^2 G}{\partial \varphi^2} \Big|_{(\theta, \pi/2)} = -4ef \sin^2(\theta) H_{\varphi=\pi/2} < 0. \quad (19)$$

Since for any θ the second derivative $\partial^2 G / \partial \varphi^2$ is always negative for $\varphi = \pi/2$, we only need to deal with the minimization problem for the case of $\varphi = 0$. To minimize $G(\theta, \varphi)$ becomes to minimize $G(\theta, 0)$ which can be written as $G(\theta, 0) = -\sum_{j,k=1}^2 w_{j,k} \log_2 w_{j,k}$ with

$$w_{j,k=1,2} = \frac{1}{4} \left(1 + (-1)^j y \cos(\theta) + (-1)^k \sqrt{t_1^2 \sin^2(\theta) + (x + (-1)^j t_3 \cos(\theta))^2} \right). \quad (20)$$

The derivative (14) is zero for either $\sin \theta = 0$ or $H_\theta = 0$. $\sin \theta = 0$ gives an extreme point $\theta = \theta_e = 0$. While $H_\theta = 0$ has the one obvious solution $\theta_e = \pi/2$ and a special solution θ_s that depends on the density matrix entries. The optimization problem is then reduced to study the second derivative of $G(\theta, 0)$ with respect to θ evaluated at the critical angles $\theta_e = 0, \pi/2$ and θ_s . Denote $H'_\theta = \partial H_\theta / \partial \theta$. The second derivative $\partial^2 G / \partial^2 \theta = -(\cos \theta H_\theta + \sin \theta H'_\theta) / 4$ evaluated at the three θ_e s depends on the behavior of two quantities,

$$\begin{aligned} H_0 &= -\partial^2 G / \partial^2 \theta|_{\theta=0} \\ &= \frac{t_1^2 - t_3|x + t_3|}{|x + t_3|} \log_2 \frac{p_+ + |x + t_3|}{p_+ - |x + t_3|} + y \log_2 \frac{p_-^2 - (x - t_3)^2}{p_+^2 - (x + t_3)^2} \\ &\quad + \frac{t_1^2 + t_3|x - t_3|}{|x - t_3|} \log_2 \frac{p_- + |x - t_3|}{p_- - |x - t_3|}, \end{aligned} \quad (21)$$

and

$$\begin{aligned} H'_{\pi/2} &= -\partial^2 G / \partial^2 \theta|_{\theta=\pi/2} \\ &= -4[(t_3^2 x^2 + (y^2 - 2t_3 xy)(t_1^2 + x^2))\sqrt{t_1^2 + x^2} \\ &\quad + t_1^2(t_1^2 + x^2 - 1)(t_1^2 - t_3^2 + x^2) \tanh^{-1} \left(\sqrt{t_1^2 + x^2} \right)] / [(t_1^2 + x^2 - 1)(t_1^2 + x^2)^{3/2}]. \end{aligned} \quad (22)$$

The sign of H_0 and $H'_{\pi/2}$ determines which of $G(0, 0)$, $G(\pi/2, 0)$ and $G(\theta_s, 0)$ is the minimum.

(1) If $H_0 > 0$ and $H'_{\pi/2} > 0$, then θ_s takes values in $(0, \pi/2)$. In this case, the minimum of $G(\theta_s, 0)$ depends on the state. For given ρ_{AB} , θ_s can be calculated numerically from $H_\theta = 0$.

(2) Otherwise, we have the minimum either $G(0, 0)$ or $G(\pi/2, 0)$,

$$G(0, 0) = - \sum_{k,j \in \{0,1\}} Q_{k,j} \log_2 Q_{k,j}, \quad (23)$$

with $Q_{k,j} = \frac{1}{4}[(1 + (-1)^k x) + (-1)^j(y + (-1)^k t_3)]$, and

$$G(\pi/2, 0) = 1 + \mathcal{L}\left(\frac{1}{2} \left(1 - \sqrt{t_1^2 + x^2} \right)\right), \quad (24)$$

where $\mathcal{L}(w) = -w \log_2 w - (1 - w) \log_2 (1 - w)$ is the binary entropy. Thus, we have the following result: one-way quantum deficit of ρ_{AB} is given by

$$\tilde{\Delta} = \sum_{\pm} (u_{\pm} \log_2 u_{\pm} + v_{\pm} \log_2 v_{\pm}) + G, \quad (25)$$

where

$$G = \begin{cases} G(\theta_s, 0), & H_0 > 0 \text{ and } H'_{\pi/2} > 0, \theta_s \in (0, \pi/2); \\ \min\{G(0, 0), G(\pi/2, 0)\}, & \text{others.} \end{cases} \quad (26)$$

By careful numerical analysis, there is at most one zero point of first derivative of $G(\theta, 0)$ with respect to θ , and only when $H_0 > 0$ and $H'_{\pi/2} > 0$, one gets the minimum inside the interval $\theta_s \in (0, \pi/2)$. Therefore, we can obtain the analytical minimum of G at $\theta = 0, \pi/2$ or $\theta = \theta_s$. In the following we present some detailed examples.

Example 1. Consider the class of special X states defined by

$$\rho_{AB} = q|\psi^-\rangle\langle\psi^-| + (1-q)|00\rangle\langle 00|, \quad (27)$$

where $|\psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$, $q \in [0, 1]$.

By using the Bloch sphere representation, from (21) and (22) we have

$$H_0(q) = \frac{(q-1)}{|2-3q|} \left[|2-3q| \log_2 \left(\frac{2}{q} - 2 \right) + (5q-2) \log_2 \left(\frac{q-2+|2-3q|}{q-2-|2-3q|} \right) \right] \quad (28)$$

and

$$H'_{\pi/2} = \frac{4(1-q)}{(2q^2-2q+1)^{3/2}} \left(\sqrt{2q^2-2q+1} (4q^2-3q+1) - 2q^3 \tanh^{-1} \left(\sqrt{2q^2-2q+1} \right) \right). \quad (29)$$

The case that both $H_0(q) > 0$ and $H'_{\pi/2}(q) > 0$ happens only in interval $q \in (1/2, 0.67)$, see Fig.1(a), where $q = 1/2$ and 0.67 are the solutions of $H_0(q) = 0$ and $H'_{\pi/2} = 0$, respectively. From (25), we have

$$\tilde{\Delta} = \begin{cases} q, & \theta = 0, q \in [0, 1/2]; \\ G(\theta_s, 0), & q \in (1/2, 0.67]; \\ q \log_2 q + (1-q) \log_2 (1-q) + 1 + \mathcal{L}\left(\frac{1+\sqrt{q^2+(1-q)^2}}{2}\right), & \theta = \pi/2, q \in (0.67, 1], \end{cases} \quad (30)$$

see Fig.1(b).

Example 2. We consider the state

$$\rho_\alpha = \alpha|\phi\rangle\langle\phi| + (1-\alpha)/2(|01\rangle\langle 01| + |10\rangle\langle 10|), \quad (31)$$

with $|\phi^+\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$.

For this case we have

$$H_0(\alpha) = \frac{2(\alpha-1)(3\alpha-1)(\log_2(1-|2\alpha-1|) - \log_2(1+|2\alpha-1|))}{|2\alpha-1|}, \quad (32)$$

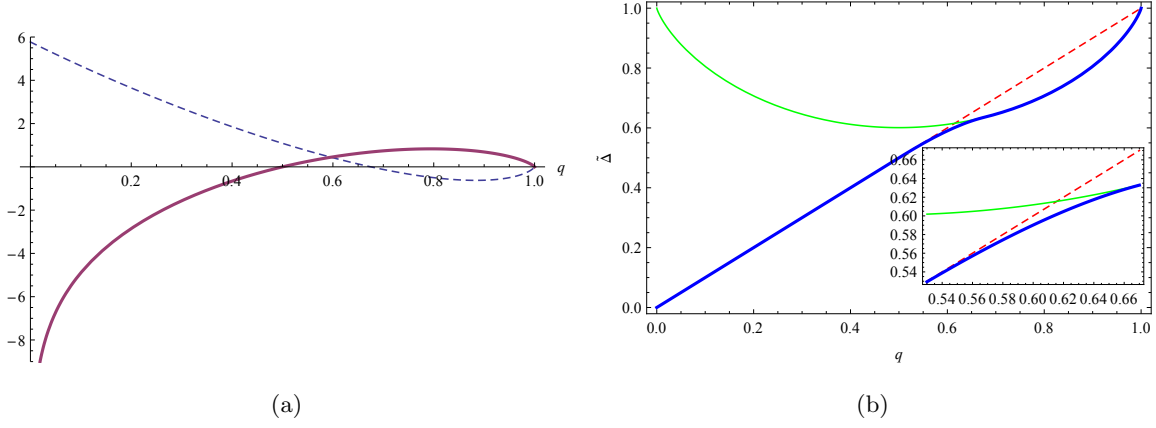


FIG. 1: (a) H_0 (purple thick solid line) and $H'_{\pi/2}$ (blue dashed line) with respect to q . (b) One-way quantum deficit (blue thick line) via q . The dashed red line stands for $G(0, 0)$ which is the one-way quantum deficit for $q \in [0, 1/2]$. The green line is for $G(\pi/2, 0)$. It coincides with the one-way quantum deficit for $q \in (0.67, 1]$.

and

$$H'_{\pi/2} = 4(\alpha - 1)(3\alpha - 1) \tanh^{-1}(\alpha)/\alpha. \quad (33)$$

For $\alpha = 1/3$ both $H_0(\alpha)$ and $H'_{\pi/2}(\alpha)$ are equal to zero. There is no domain of α such that both $H_0 > 0$ and $H'_{\pi/2} > 0$, see Fig. 2(a). Hence we can easily obtain the analytical expression of one-way quantum deficit for ρ_α ,

$$\tilde{\Delta} = \begin{cases} \alpha, & \theta = 0, \alpha \in [0, 1/3]; \\ 1 + \alpha + \alpha \log_2 \alpha + \frac{1}{2} [(1 - \alpha) \log_2(1 - \alpha) - (1 + \alpha) \log_2(1 + \alpha)], & \theta = \pi/2, \alpha \in (1/3, 1], \end{cases} \quad (34)$$

see Fig.2(b) for the analytical expression of one-way quantum deficit vs α .

Example 3. Consider the Bell diagonal state,

$$\rho_{AB} = \frac{1}{4} [I \otimes I + \sum_{i=1}^3 t_i \sigma_i \otimes \sigma_i]. \quad (35)$$

We have

$$H_0 = \frac{2(t_1^2 - t_3^2)}{|t_3|} \log_2 \left(\frac{1 + |t_3|}{1 - |t_3|} \right) \quad (36)$$

and

$$H'_{\pi/2} = -4(t_1^2 - t_3^2) \tanh^{-1}(|t_1|)/|t_1|. \quad (37)$$

Since $\log_2 \frac{1+|t_3|}{1-|t_3|} > 0$ and $\tanh^{-1}(|t_1|) > 0$, H_0 and $H'_{\pi/2}$ cannot be greater than zero simultaneously.

We obtain the analytical expression $\min\{G(0, 0), G(\pi/2, 0)\}$,

$$G(0, 0) = 1 + \mathcal{L}\left(\frac{1 + |t_3|}{2}\right) \quad (38)$$

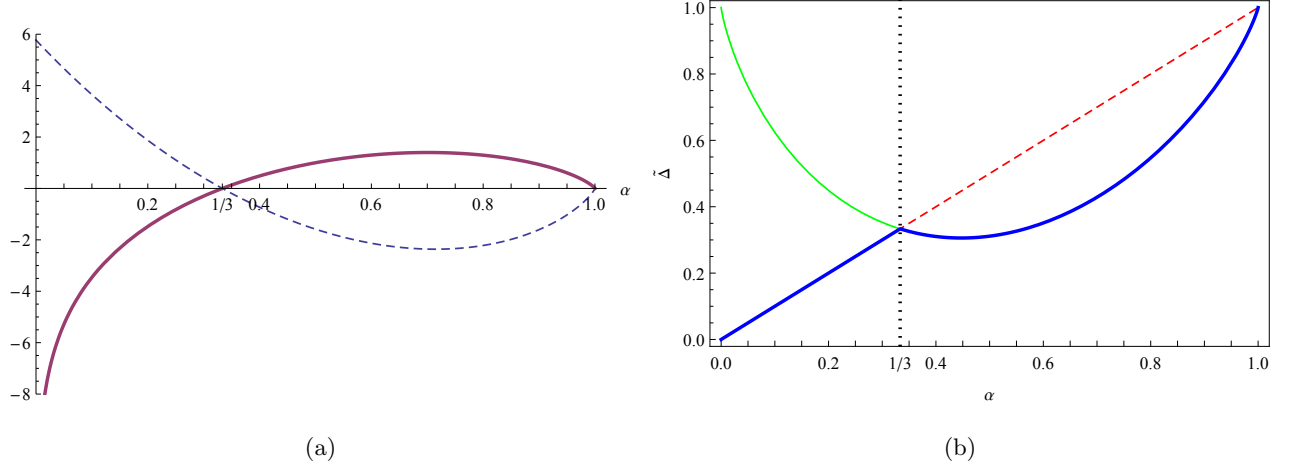


FIG. 2: (a) H_0 (purple thick solid line) and $H'_{\pi/2}$ (blue dashed line) with respect to α . (b) One-way quantum deficit (blue thick line) via α . Dashed red line for G at $\theta = 0$, and green line for G at $\theta = \pi/2$.

and

$$G(\pi/2, 0) = 1 + \mathcal{L}\left(\frac{1 + |t_1|}{2}\right). \quad (39)$$

Therefore $\min\{G(0, 0), G(\pi/2, 0)\} = 1 + \mathcal{L}(\frac{1+t}{2})$, where $t = \max\{|t_1|, |t_3|\}$. In fact, for Bell-diagonal states, the optimization is obtained at $t_3 = \pm t_1$ or $t_3 = \pm t_2$ [22]. Therefore, we recovered the result in Ref.[31] where $t = \max\{|t_1|, |t_2|, |t_3|\}$.

III. ONE-WAY QUANTUM DEFICIT UNDER DECOHERENCE

A system undergoes environmental noises can be characterized by Kraus operators. We consider quantum two-qubit systems subjecting to dephasing channels described by the Kraus operators $F_0 = |0\rangle\langle 0| + \sqrt{1-\gamma}|1\rangle\langle 1|$ and $F_1 = \sqrt{\gamma}|1\rangle\langle 1|$, where $\gamma = 1 - e^{-\tau t}$ and τ is phase damping rate [37]. Under the channel the ρ_{AB} is changed to be

$$\begin{aligned} \rho'_{AB} &= \sum_{i,j=0}^1 F_i^A \otimes F_j^B \rho_{AB} (F_i^A \otimes F_j^B)^\dagger \\ &= \frac{1}{4} [I \otimes I + x\sigma_3 \otimes I + yI \otimes \sigma_3 + \sum_{i=1}^2 (1-\gamma)^2 t_i \sigma_i \otimes \sigma_i + (t_3 \sigma_3 \otimes \sigma_3)]. \end{aligned}$$

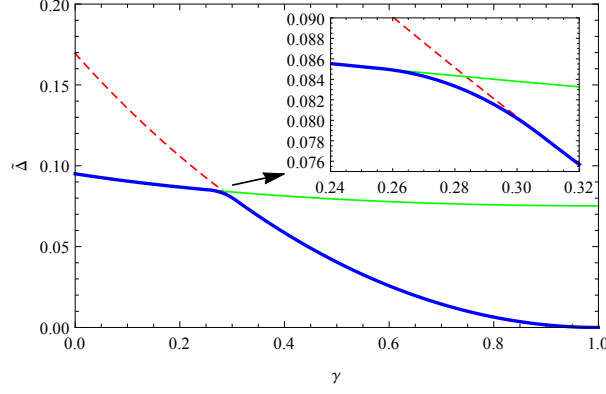


FIG. 3: One-way quantum deficit vs γ under dephasing noise for $x = 0.45$, $y = 0.32$, $t_1 = 0.43$, $t_2 = 0.09$ and $t_3 = 0.15$.

We see that t_1 and t_2 have been transformed to $(1 - \gamma)^2 t_1$ and $(1 - \gamma)^2 t_2$. We have

$$\begin{aligned}
 H_0 &= -\partial^2 G / \partial^2 \theta|_{\theta=0} \\
 &= \frac{\Upsilon^2 - t_3|x + t_3|}{|x + t_3|} \log_2 \frac{p_+ + |x + t_3|}{p_+ - |x + t_3|} + y \log_2 \frac{p_-^2 - (x - t_3)^2}{p_+^2 - (x + t_3)^2} \\
 &\quad + \frac{\Upsilon^2 + t_3|x - t_3|}{|x - t_3|} \log_2 \frac{p_- + |x - t_3|}{p_- - |x - t_3|}
 \end{aligned} \tag{40}$$

and

$$\begin{aligned}
 H'_{\pi/2} &= \Upsilon^2 (\Upsilon^2 + x^2 - 1) (\Upsilon^2 - t_3^2 + x^2) \tanh^{-1} \left(\sqrt{\Upsilon^2 + x^2} \right) / ((\Upsilon^2 + x^2 - 1) (\Upsilon^2 + x^2)^{3/2}) \\
 &\quad - 4[(-2t_3xy(\Upsilon^2 + x^2) + x^2y^2 + \Upsilon^2y^2 + t_3^2x^2)\sqrt{\Upsilon^2 + x^2}],
 \end{aligned} \tag{41}$$

where $\Upsilon = (1 - \gamma)^2 t_1$. It is direct to verify that $G(0, 0)$ exactly given by (23). While $G(\pi/2, 0)$ has the following form,

$$G(\pi/2, 0) = 1 + \mathcal{L}\left(\frac{1}{2} \left(1 - \sqrt{\Upsilon^2 + x^2}\right)\right). \tag{42}$$

As an example, taking $x = 0.45$, $y = 0.32$, $t_1 = 0.43$, $t_2 = 0.09$ and $t_3 = 0.15$, we can observe the one-way quantum deficit under phase damping channel, see Fig. (3). It should be emphasized that the exact boundaries exist between three different branches and sudden transitions occur in the phase damping channel.

IV. SUMMARY

We have provided an effective approach to get analytical results of one-way quantum deficit for general two-qubit X states. Analytical formulae of one-way quantum deficit have been obtained

in general for states such that $\min\{G(0,0), G(\pi/2,0)\}$. It has been shown that only in very few cases, the conditions $H_0 > 0$ and $H'_{\pi/2} > 0$ are satisfied. Even in such special cases, the one-way quantum deficit can be easily calculated by solving θ_s from a one-parameter equation. We have also studied the decoherence of one-way quantum deficit under phase damping channel. As for a detailed example, it has been shown that the one-way quantum deficit changes gradually phase damping channel. There is no behavior like sudden death of quantum entanglement.

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